First-Order Logic with Connectivity Operators

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First-order logic (FO) can express many algorithmic problems on graphs, such as the independent set and dominating set problem parameterized by solution size. On the other hand, FO cannot express the very simple algorithmic question whether two vertices are connected. We enrich FO with connectivity predicates that are tailored to express algorithmic graph problems that are commonly studied in parameterized algorithmics. By adding the atomic predicates $\mathrm{conn}_k(x, y, z_1, \ldots, z_k)$ that hold true in a graph if there exists a path between (the valuations of) x and y after (the valuations of) $z_1, \ldots, z_k$ have been deleted, we obtain *separator logic* $\mathrm{FO} + \mathrm{conn}$. We show that separator logic can express many interesting problems such as the feedback vertex set problem and elimination distance problems to first-order definable classes. Denote by $\mathrm{FO} + \mathrm{conn}_k$ the fragment of separator logic that is restricted to connectivity predicates with at most $k + 2$ variables (that is, at most k deletions), we show that $\mathrm{FO} + \mathrm{conn}_{k+1}$ is strictly more expressive than $\mathrm{FO} + \mathrm{conn}_k$ for all $k \geq 0$. We then study the limitations of separator logic and prove that it cannot express planarity, and, in particular, not the disjoint paths problem. We obtain the stronger *disjoint-paths logic* $\mathrm{FO} + \mathrm{DP}$ by adding the atomic predicates $\text{disjoint-paths}_k[(x_1, y_1), \ldots, (x_k, y_k)]$ that evaluate to true if there are internally vertex-disjoint paths between (the valuations of) x_i and y_i for all $1 \leq i \leq k$. Disjoint-paths logic can express the disjoint paths problem, the problem of (topological) minor containment, the problem of hitting (topological) minors, and many more. Again we show that the fragments $\mathrm{FO} + \mathrm{DP}_k$ that use predicates for at most k disjoint paths form a strict hierarchy of expressiveness. Finally, we compare the expressive power of the new logics with that of transitive-closure logics and monadic second-order logic.

CCS Concepts: • **Theory of computation → Finite Model Theory**; • **Mathematics of computing → Combinatorics**.

Additional Key Words and Phrases: first-order logic, graph theory, connectivity

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1 INTRODUCTION

Logic provides a very elegant way of formally describing computational problems. Fagin's celebrated result from 1974 [17] established that existential second-order logic captures the complexity class NP. Fagin thereby provided a machine-independent characterization of a complexity class and initiated the field of descriptive complexity theory. Many other complexity classes were later characterized by logics in this theory. Today it remains one of the major open problems whether there exists a logic capturing PTime.

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In 1990 Courcelle proved that every graph problem definable in monadic second-order logic (MSO) can be decided in linear time on graphs of bounded treewidth [10]. This theorem has a much more algorithmic (rather than a complexity-theoretic) flavor, in the sense that, from a logical description of a problem, it derives an algorithmic approach on how to solve it on certain graph classes. Grohe in his seminal survey coined the term *algorithmic meta-theorem* for such theorems that provide general conditions on a problem and on the input instances that, when satisfied, imply the existence of an efficient algorithm for the problem [25]. Courcelle's theorem for MSO was extended to graph classes with bounded cliquewidth [11] and it is known that these are essentially the most general graph classes on which efficient MSO model checking [22, 29] is possible. MSO is a powerful logic that can express many important algorithmic problems on graphs. With quantification over edges, we can for example express the existence of a Hamiltonian path, the existence of a fixed minor or topological minor, the disjoint paths problem, and many deletion problems. For a property Π, the task in the Π-deletion problem is to find in a given graph G a minimum-size subset S of $V(G)$ such that the graph $G - S$ obtained from G by removing S has the property Π. Important examples of Π-deletion problems are the feedback vertex set problem, the odd cycle transversal problem, or the problem of hitting all minors or topological minors from a given list $\mathcal{F}$. We refer to [13] for the formal definitions of the mentioned algorithmic problems. Also, many elimination distance problems recently studied [6] in parameterized algorithmics can be expressed in MSO. However, as we have seen, this expressiveness comes at the price of algorithmic intractability already on very restricted graph classes. This cannot be a surprise as e.g. the Hamiltonian path problem is NP-complete already on planar graphs of maximum degree 3 [7].

First-order logic (FO) is much weaker than MSO and not surprisingly, the model checking problem can be solved efficiently on much more general graph classes. FO model checking is fixed-parameter tractable on a subgraph-closed class $\mathscr{C}$ if and only if $\mathscr{C}$ is nowhere dense [26] and a recent breakthrough result showed that it is fixed-parameter tractable on a class $\mathscr{C}$ of ordered graphs if and only if $\mathscr{C}$ has bounded twin-width [4]. FO is weaker than MSO but it can still express many important problems such as the independent set problem and dominating set problem parameterized by solution size, the Steiner tree problem parameterized by the number of Steiner vertices, and many more problems. On the other hand, first-order logic cannot even express the algorithmically extremely simple problem of whether a graph is connected. Also, the other algorithmic problems mentioned before are not expressible in FO, even though some of them are fixed-parameter tractable on general graphs. For example, we can efficiently test for a fixed minor or topological minor and solve the disjoint paths problem [36]. Many Π-deletion problems are fixed-parameter tractable, see e.g. [12, 20, 33], as well as many elimination distance problems [1, 18].

The fact that first-order logic can only express local problems is classically addressed by adding transitive-closure or fixed-point operators, see e.g. [16, 24, 30]. Unfortunately, this again comes at the price of intractable model checking for very restricted graph classes. For example, even the model checking problem for the very restricted monadic transitive-closure logic TC^1 studied by Grohe [25], is $\mathrm{AW}[\star]$-hard on planar graphs of maximum degree at most 3 [25, Theorem 7.3]. Extensions of first-order logic with a reachability predicate or with predicates for reachability with an additional regular expression (over labeled transitions) are studied for example in [9, 15, 38]. These extensions play an important role for specification in system analysis, as they can express safety and liveness conditions (in transition systems). The main focus of study for these latter logics are questions of decidability. Furthermore, they fall short of being able to express the above mentioned algorithmic graph problems.

This motivates our present work in which we enrich first-order logic with more powerful connectivity predicates. The extensions are tailored to express algorithmic graph problems that are studied in recent parameterized algorithmics. Adding the atomic predicate $\mathrm{conn}_0(x, y)$ that evaluates to true on a graph G if (the valuations of) x and y are connected in G yields the mentioned extension of first-order logic with a reachability predicate. This predicate easily generalizes to directed graphs but for simplicity, we work with undirected graphs only. Of course, with this predicate we can express connectivity of graphs, however, it falls short of expressing other interesting graph problems, e.g. it cannot express that a graph is acyclic. We hence introduce more general predicates $\mathrm{conn}_k(x, y, z_1, \ldots, z_k)$, parameterized by a number k, that evaluate to true on a graph G if (the valuations of) x and y are connected in G once (the valuations of) $z_1, \ldots, z_k$ have been deleted. The interplay of these predicates with the usual nesting of first-order quantification makes the new logic FO + conn already quite powerful. For example, we can express simple graph problems such as 2-connectivity by $\forall z \forall x \forall y \big(x \neq z \wedge y \neq z \rightarrow \mathrm{conn}_1(x, y, z)\big)$. We can also express many deletion problems, such as the feedback vertex set problem, and the elimination distance to bounded degree, and more generally, elimination distance to any fixed first-order property.

We also point to Mikołaj Bojańczyk's work [3], who independently introduced FO + conn and proposed the name *separator logic*. He studied a variant of star-free expressions for graphs and showed that these two formalisms for defining graph languages are equivalent. We follow his suggestion for the name of the new logic and thank Mikołaj for the discussion on separator logic.

In Section 3 we study the expressive power of separator logic. We give examples of problems expressible with separator logic as well as proofs that certain problems, such as planarity and in particular the disjoint paths problem, are not expressible in separator logic. We show that $(k + 2)$-connectivity of a graph cannot be expressed with only conn_k predicates and conclude that the restricted use of these predicates induces a natural hierarchy of expressiveness.

The fact that planarity and the disjoint paths problem cannot be expressed in separator logic motivates us to define an even stronger logic that can express these problems. We define atomic predicates of the form $\text{disjoint-paths}_k[(x_1, y_1), \ldots, (x_k, y_k)]$ that evaluate to true if and only if there are k internally vertex-disjoint paths between (the valuations of) x_i and y_i for all $1 \leq i \leq k$. Connectivity of x and y can be tested by $\text{disjoint-paths}_1[(x, y)]$. More generally, the so obtained *disjoint-paths logic* FO + DP strictly extends separator logic. With this more powerful logic, we can test if a graph contains a fixed minor or topological minor, and in particular, test for planarity. In combination with first-order quantification, we can also express many Π-deletion problems such as the problem of hitting all minors or topological minors from a given list $\mathcal{F}$. On the other hand, we cannot express the odd cycle transversal problem, as we cannot even express bipartiteness of a graph. We study the expressive power of FO + DP in Section 4. Among other results, we prove that again an increase in the number of disjoint paths in the predicates leads to an increase in expressive power.

Note that while it would be desirable to be able to express bipartiteness, which is equivalent to 2-colorability, it is not desirable to express general colorability problems, as we aim for logics that are tractable on planar graphs and beyond, while the 3-colorability problem is NP-complete on planar graphs. This example shows again that it is a delicate balance between expressiveness and tractability and it will be a challenging and highly interesting problem in future work to find the right set of predicates to express even more algorithmic graph problems while at the same time having tractable model checking.

We conclude the paper in Section 5 with a comparison between the newly introduced logics and more established ones, like MSO and transitive-closure logics.

2 PRELIMINARIES

Graphs. In this paper, we deal with finite and simple undirected graphs. Let G be a graph. We write $V(G)$ for the vertex set of G and $E(G)$ for its edge set. For a set $X \subseteq V(G)$ we write $G[X]$ for the subgraph of G induced by X and $G - X$ for the subgraph induced by $V(G) \setminus X$. For a singleton set $\{v\}$ we write $G - v$ instead of $G - \{v\}$. A *path* P in G is a subgraph on distinct vertices $v_1, \ldots, v_t$ with $\{v_i, v_{i+1}\} \in E(P)$ for all $1 \leq i < t$ and a path P is said to *connect* its endpoints v_1 and v_t. Two paths are *internally vertex-disjoint* if and only if every vertex that appears in both paths is an endpoint of both paths. The graph G is *connected* if every two of its vertices are connected by a path. It is *k-connected* if G has more than k vertices and $G - X$ is connected for every subset

$X \subseteq V(G)$ of size strictly smaller than k. A *cycle* C in G is a subgraph on distinct vertices $v_1, \dots, v_t$, $t \geq 3$, with $\{v_i, v_{i+1}\} \in E(C)$ for all $1 \leq i < t$ and $\{v_t, v_1\} \in E(C)$. An acyclic graph is a *forest* and a connected acyclic graph is a *tree*.

A graph H is a *minor* of G, denoted $H \preccurlyeq G$, if for all $v \in V(H)$ there are pairwise vertex-disjoint connected subgraphs G_v of G such that whenever $\{u, v\} \in E(H)$, then there are $x \in V(G_u)$ and $y \in V(G_v)$ with $\{x, y\} \in E(G)$. The subgraph G_v is called the *branch set of v in G*. The graph H is a *topological minor* of G, denoted $H \preccurlyeq^{top} G$, if for all $v \in V(H)$ there is a distinct vertex x_v in G and for all $\{u, v\} \in E(H)$ there are internally vertex-disjoint paths P_{uv} in G with endpoints x_u and x_v. The vertices x_v are called the *principal vertices* of the topological minor model of H in G. A graph is *planar* if and only if it contains neither K_5, the complete graph on 5 vertices, nor $K_{3,3}$, the complete bipartite graph with two partitions of size 3, as a minor [40].

Logic. In this work, we deal with structures over purely relational *signatures*. A (purely relational) signature is a collection of relation symbols, each with an associated arity. Let σ be a signature. A *σ-structure* $\mathfrak{A}$ consists of a non-empty set A, the universe of $\mathfrak{A}$, together with an interpretation of each k-ary relation symbol $R \in \sigma$ as a k-ary relation $R^{\mathfrak{A}} \subseteq A^k$. For a subset $X \subseteq A$ we write $\mathfrak{A}[X]$ for the substructure induced by X. A *partial isomorphism* between σ-structures $\mathfrak{A}$ and $\mathfrak{B}$ is an isomorphism between $\mathfrak{A}[X]$ and $\mathfrak{B}[Y]$ for some subset $X \subseteq A$ of the universe A of $\mathfrak{A}$ and some subset $Y \subseteq B$ of the universe B of $\mathfrak{B}$.

We assume an infinite supply $\textsc{Var}$ of variables. First-order σ-formulas are built from the atomic formulas $x = y$, where x and y are variables, and $R(x_1, \dots, x_k)$, where $R \in \sigma$ is a k-ary relation symbol and $x_1, \dots, x_k$ are variables, by closing under the Boolean connectives $\neg$, $\wedge$ and $\vee$, and by existential and universal quantification $\exists x$ and $\forall x$. A variable x not in the scope of a quantifier is a *free variable*. A formula without free variables is a *sentence*. The *quantifier rank* $\mathrm{qr}(\varphi)$ of a formula φ is the maximum nesting depth of quantifiers in φ. We write $\mathrm{FO}_\sigma[q]$ for the set of all FO σ-formulas of quantifier rank at most q, or simply $\mathrm{FO}[q]$ if σ is clear from the context. A formula without quantifiers is called *quantifier-free*.

If $\mathfrak{A}$ is a σ-structure with universe A, then an *assignment* of the variables in $\mathfrak{A}$ is a mapping $\bar{a} : \textsc{Var} \to A$. We use the standard notation $(\mathfrak{A}, \bar{a}) \models \varphi(\bar{x})$ or $\mathfrak{A} \models \varphi(\bar{a})$ to indicate that φ is satisfied in $\mathfrak{A}$ when the free variables $\bar{x}$ of φ have been assigned by $\bar{a}$. We refer e.g. to the textbook [30] for more background on first-order logic.

3 SEPARATOR LOGIC

In this section, we study the expressive power of separator logic FO + conn. Formally, we assume that σ is a signature that does not contain any of the relation symbols conn_k for all $k \geq 0$, and that it does contain a binary relation symbol E, representing an edge relation. We assume that E is always interpreted as an irreflexive and symmetric relation and connectivity will always refer to this relation. We let $\sigma + \mathrm{conn} := \sigma \cup \{\mathrm{conn}_k : k \geq 0\}$, where each conn_k is a $(k+2)$-ary relation symbol.

Definition 3.1. The formulas of $(\mathrm{FO} + \mathrm{conn})_\sigma$ are the formulas of $\mathrm{FO}_{\sigma+\mathrm{conn}}$. We usually simply write FO + conn, when σ is understood from the context.

For a σ-structure $\mathfrak{A}$, an assignment $\bar{a}$ and an FO + conn formula $\varphi(\bar{x})$, we define the satisfaction relation $(\mathfrak{A}, \bar{a}) \models \varphi(\bar{x})$ as for first-order logic, where an atomic predicate $\mathrm{conn}_k(x, y, z_1, \dots, z_k)$ is evaluated as follows. Assume that the universe of $\mathfrak{A}$ is A and let $G = (A, E^{\mathfrak{A}})$ be the graph on vertex set A and edge set $E^{\mathfrak{A}}$. Then $(\mathfrak{A}, \bar{a})$ is a model of $\mathrm{conn}_k(x, y, z_1, \dots, z_k)$ if and only if $\bar{a}(x)$ and $\bar{a}(y)$ are connected in $G - \{\bar{a}(z_1), \dots, \bar{a}(z_k)\}$.

Note in particular that if $\bar{a}(x) = \bar{a}(z_i)$ or $\bar{a}(y) = \bar{a}(z_i)$ for some $i \leq k$, then $(\mathfrak{A}, \bar{a}) \not\models \mathrm{conn}_k(x, y, z_1, \dots, z_k)$.

We write $\mathrm{FO} + \mathrm{conn}_k$ for the fragment of FO + conn that uses only conn_ℓ predicates for $\ell \leq k$. The quantifier rank of an FO + conn formula is defined as for plain first-order logic. For structures $\mathfrak{A}$ with universe A and $\bar{a} \in A^m$ and $\mathfrak{B}$ with universe B and $\bar{b} \in B^m$, we write $(\mathfrak{A}, \bar{a}) \equiv_{\mathrm{conn}} (\mathfrak{B}, \bar{b})$ if $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{B}, \bar{b})$ satisfy the same FO + conn formulas, that is, for all $\varphi(\bar{x}) \in$ FO + conn we have $\mathfrak{A} \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{b})$. Similarly, we write

$(\mathfrak{A},\bar{a}) \equiv_{\mathrm{conn}_k} (\mathfrak{B},\bar{b})$ and $(\mathfrak{A},\bar{a}) \equiv_{\mathrm{conn}_{k,q}} (\mathfrak{B},\bar{b})$ if $(\mathfrak{A},\bar{a})$ and $(\mathfrak{B},\bar{b})$ satisfy the same $\mathrm{FO}+\mathrm{conn}_k$ formulas and the same $\mathrm{FO}+\mathrm{conn}_k$ formulas of quantifier rank at most q, respectively.

3.1 Expressive power of separator logic

We now give examples of graph problems that are expressible with separator logic.

Example 3.2. Connectivity is expressible in $\mathrm{FO}+\mathrm{conn}_0$ by the formula

$$\forall x \forall y \big(\mathrm{conn}_0(x,y)\big).$$

More generally, for every non-negative integer k, $(k+1)$-connectivity can be expressed by the formula

$$\forall x \forall y \forall z_1 \ldots \forall z_k \big(\bigwedge_{1\le i\le k} (x \neq z_i \wedge y \neq z_i) \rightarrow \mathrm{conn}_k(x,y,z_1,\ldots,z_k)\big).$$

Example 3.3. We can express that there exists a cycle by

$$\exists x \exists y \big(E(x,y) \wedge \exists z \big(\mathrm{conn}_1(z,x,y) \wedge \mathrm{conn}_1(z,y,x)\big)\big),$$

hence, that a graph is acyclic by the negation of that formula. We write $\psi_{acyclic}$ for that formula. We can express that a graph is a tree by stating that it is connected and acyclic.

We can conveniently express deletion problems by relativizing formulas as follows. For a formula φ that does not contain z as a free variable write $\mathrm{del}(z)[\varphi]$ for the formula obtained from φ by recursively replacing every subformula $\exists x\psi$ by $\exists x(x \neq z \wedge \psi)$, every subformula $\forall x\psi$ by $\forall x(x \neq z \rightarrow \psi)$ and every atomic formula $\mathrm{conn}_k(x,y,z_1,\ldots,z_k)$ by $\mathrm{conn}_{k+1}(x,y,z_1,\ldots,z_k,z)$. Then $(\mathfrak{A},\bar{a}) \models \mathrm{del}(z)[\varphi]$ if and only if $(\mathfrak{A}-\bar{a}(z),\bar{a}) \models \varphi$, where $\mathfrak{A}-\bar{a}(z)$ denotes the substructure induced on the universe of $\mathfrak{A}$ without $\bar{a}(z)$.

Example 3.4. We can state the existence of a feedback vertex set of size k by

$$\exists z_1 \mathrm{del}(z_1)[\cdots[\exists z_k \mathrm{del}(z_k)[\psi_{acyclic}]\ldots].$$

We can of course use the same principle to express any Π-deletion problem that is $\mathrm{FO}+\mathrm{conn}$ expressible.

We can also express that a formula φ holds in a connected component.

Example 3.5. We write $\mathrm{comp}(x)$ for the connected component of (the valuation of) x. For a formula φ we write $\varphi^{[\mathrm{comp}(x)]}$ for the formula obtained from φ by recursively replacing all subformulas $\exists y\psi$ by $\exists y(\mathrm{conn}_0(x,y)\wedge\psi)$ and all subformulas $\forall y\psi$ by $\forall y(\mathrm{conn}_0(x,y) \rightarrow \psi)$. Then $(\mathfrak{A},\bar{a}) \models \varphi^{[\mathrm{comp}(x)]}$ if and only if $(\mathfrak{A}[\mathrm{comp}(\bar{a}(x))],\bar{a}) \models \varphi$, where $\mathfrak{A}[\mathrm{comp}(\bar{a}(x))]$ denotes the substructure induced on the connected component of $\bar{a}(x)$.

Using this relativization to connected components, we can also express many elimination distance problems.

Example 3.6. The *elimination distance* to a class $\mathscr{C}$ of graphs measures the number of recursive deletions of vertices needed for a graph G to become a member of $\mathscr{C}$. More precisely, a graph G has elimination distance 0 to $\mathscr{C}$ if $G \in \mathscr{C}$, and otherwise elimination distance at most $k+1$ if in every connected component of G we can delete a vertex such that the resulting graph has elimination distance at most k to $\mathscr{C}$. Elimination distance was introduced by Bulian and Dawar [6] in their study of the parameterized complexity of the graph isomorphism problem and has recently obtained much attention in the literature, see e.g. [1, 5, 19, 27, 28, 31].

Now assume $\mathscr{C}$ is a first-order definable class, say defined by a formula $\psi_{\mathscr{C}}$. Then elimination distance 0 to $\mathscr{C}$ is defined by $\mathrm{ed}_0 = \psi_{\mathscr{C}}$. If ed_k has been defined, then we can express elimination distance $k+1$ to $\mathscr{C}$ by the formula

$$\mathrm{ed}_{k+1} \coloneqq \mathrm{ed}_k \vee \forall x \big(\exists y\ \mathrm{del}(y)[\mathrm{ed}_k]\big)^{[\mathrm{comp}(x)]}.$$

Our final example concerns the expressive power of separator logic on finite words and finite trees. By the classical result of Büchi [8], a language on words is regular if and only if it is definable in MSO. Here, words are represented as finite structures over the vocabulary of the successor relation and unary predicates representing the letters of the alphabet. When considering first-order logic on strings, it makes a big difference whether one considers word structures over the successor relation or over its transitive closure, the order relation. Languages definable by FO over the order relation are exactly the star-free languages (see e.g. [30, Theorem 7.26]), while languages definable by FO over the successor relation are exactly the locally threshold testable languages [39, Theorem 4.8]. Similarly, MSO on trees can define exactly the regular tree languages (defined via tree automata, see [30, Theorem 7.30]), while FO can only define a proper subclass of the regular tree languages when the ancestor-descendant or even only the parent-child relation is present. This background was also Bojańczyk's motivation, who studied a variant of star-free expressions for graphs and showed that these two formalisms for defining graph languages are equivalent [3]. In our example, we show that separator logic on rooted trees has exactly the same expressive power as first-order logic in the presence of the ancestor-descendant relation. Let us write FO$[<]$ for the latter logic. On the other hand, we treat a rooted tree as a graph-theoretic tree with an additional unary predicate marking the root. In the degenerate case, we treat a word as a path, where one of the endpoints is marked by a unary predicate as the smallest vertex (the beginning of the word).

Example 3.7. On rooted trees (and similarly on words) FO + conn collapses to FO + conn_1 and has exactly the same expressive power as FO$[<]$ over trees with the ancestor-descendant relation. We show first that $\mathrm{conn}_k(x, y, z_1, \ldots, z_k)$ can be expressed in FO$[<]$. For this, we need to ensure that x and y are not equal to any z_i and that no z_i lies on the unique path between x and y in the tree. We can define the vertices on the unique path between x and y by first defining the least common ancestor of x and y by the formula $\mathrm{lca}(x, y, z) = z \leq x \wedge z \leq y \wedge \neg\exists z'(z < z' \wedge z' \leq x \wedge z' \leq y)$. If z is the least common ancestor of x and y, it remains to state that none of the z_i lies either between x and z or between y and z, which is done by the formula $\exists z\big(\mathrm{lca}(x, y, z) \wedge \bigwedge_{1\leq i\leq k} \neg(z \leq z_i \leq x \vee z \leq z_i \leq y)\big)$.

Conversely, we show that we can define with FO + conn_1 the ancestor-descendant relation in rooted trees. Assume the root is marked by the unary symbol R. Then $x < y$ is equivalent to
$\exists r\big(R(r) \wedge \mathrm{conn}_1(x, r, y) \wedge \neg\mathrm{conn}_1(y, r, x)\big)$.

3.2 The limits of separator logic

We now study the limits of separator logic and show that planarity cannot be expressed in FO + conn. Slightly abusing notation, let us also write FO + conn_k for the problems that are expressible in FO + conn_k. We also show that there is a strict hierarchy of expressiveness: FO + $\mathrm{conn}_0 \subsetneq$ FO + $\mathrm{conn}_1 \subsetneq$ FO + $\mathrm{conn}_2 \subsetneq \ldots$ These results are based on an adaptation of the standard Ehrenfeucht-Fraïssé game (EF game), which is commonly used in the study of the expressive power of first-order logic.

Ehrenfeucht-Fraïssé Games. The Ehrenfeucht-Fraïssé game is played by two players called *Spoiler* and *Duplicator*. Given two structures $\mathfrak{A}$ and $\mathfrak{B}$, Spoiler's aim is to show that the structures can be distinguished by first-order logic (with formulas of a given quantifier rank), while Duplicator wants to prove the opposite. The q-round EF game proceeds in q rounds, where each round consists of the following two steps.

(1) Spoiler picks an element $a \in \mathfrak{A}$ or an element $b \in \mathfrak{B}$.
(2) Duplicator responds by picking an element of the other structure, that is, she picks a $b \in \mathfrak{B}$ if Spoiler chose $a \in \mathfrak{A}$, and she picks an $a \in \mathfrak{A}$ if Spoiler chose $b \in \mathfrak{B}$.

After q rounds, the game stops. Assume the players have chosen $\bar{a} = a_1, \ldots, a_q$ and $\bar{b} = b_1, \ldots, b_q$. Then Duplicator *wins* if the mapping $a_i \mapsto b_i$ for all $1 \leq i \leq q$ is a partial isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$. We write for short

$\bar{a} \mapsto \bar{b}$ for this mapping. Otherwise, Spoiler wins. We say that Duplicator *wins the q-round EF game* on $\mathfrak{A}$ and $\mathfrak{B}$ if she can force a win no matter how Spoiler plays. We then write $\mathfrak{A} \simeq_q \mathfrak{B}$.

Theorem 3.8 (Ehrenfeucht-Fraïssé, see e.g. [30, Theorem 3.18]). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two σ-structures where σ is purely relational. Then $\mathfrak{A} \equiv_q \mathfrak{B}$ if and only if $\mathfrak{A} \simeq_q \mathfrak{B}$.*

As $(\mathrm{FO} + \mathrm{conn})_\sigma$ is defined as $\mathrm{FO}_{\sigma+\mathrm{conn}}$, the EF game for FO naturally extends to separator logic. The *$(\mathrm{conn}_{k,q})$-game* is played just as the q-round EF game, where the winning condition is adapted as follows. If in q rounds the players have chosen $\bar{a} = a_1, \dots, a_q$ and $\bar{b} = b_1, \dots, b_q$, then Duplicator wins if

(1) the mapping $\bar{a} \mapsto \bar{b}$ is a partial isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$, and
(2) for every $\ell \leq k$ and every sequence $(i_1, \dots, i_{\ell+2})$ of numbers in $\{1, \dots, q\}$ we have

$$\mathfrak{A} \models \mathrm{conn}_\ell(a_{i_1}, \dots, a_{i_{\ell+2}}) \iff \mathfrak{B} \models \mathrm{conn}_\ell(b_{i_1}, \dots, b_{i_{\ell+2}}).$$

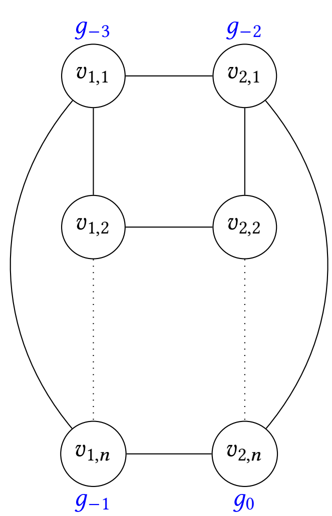
Otherwise, Spoiler wins. We say that Duplicator wins the $(\mathrm{conn}_{k,q})$-game on $\mathfrak{A}$ and $\mathfrak{B}$ if she can force a win no matter how Spoiler plays. We then write $\mathfrak{A} \simeq_{\mathrm{conn}_{k,q}} \mathfrak{B}$.

We obtain the following theorem.

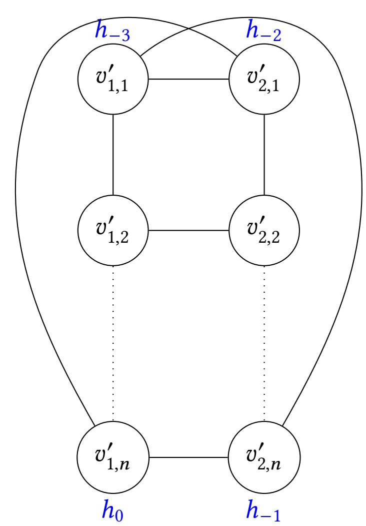
Theorem 3.9. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two σ-structures where σ is purely rational (and contains a binary relation symbol E that is interpreted on both structures as an irreflexive and symmetric relation). Then $\mathfrak{A} \equiv_{\mathrm{conn}_{k,q}} \mathfrak{B}$ if and only if $\mathfrak{A} \simeq_{\mathrm{conn}_{k,q}} \mathfrak{B}$.*

The next theorem exemplifies the use of the $(\mathrm{conn}_{k,q})$-game.

Theorem 3.10. *Planarity is not expressible in* FO + conn.



(a) G_q



(b) H_q

Fig. 1. Planarity is not expressible in FO + conn.

Proof. Assume planarity is expressible by a sentence φ of $\mathrm{FO} + \mathrm{conn}_k$ of quantifier rank q. Without loss of generality, we may assume that $k \leq q$, as otherwise, we have repetitions in the conn_k predicates that can be avoided by using conn_ℓ predicates for $\ell < k$. Let G_q and H_q be defined as shown in Figure 1, where $n = 2^{q+1}$. Then, G_q is planar but H_q contains $K_{3,3}$ as a minor and hence is not planar (it embeds only in a surface of genus one; the Möbius strip, which cannot be embedded into the plane). We show that $G_q \simeq_{\mathrm{conn}_{k,q}} H_q$, contradicting the assumption that φ must distinguish G_q and H_q. In fact, we prove an even stronger statement by giving Spoiler four free moves $g_{-3} = v_{1,1}$, $g_{-2} = v_{2,1}$, $g_{-1} = v_{1,n}$ and $g_0 = v_{2,n}$ in G_q where Duplicator responds with the vertices $h_{-3} = v'_{1,1}$, $h_{-2} = v'_{2,1}$, $h_{-1} = v'_{2,n}$ and $h_0 = v'_{1,n}$ in H_q. Note the twist in the last two vertices. Even though Duplicator's answers are forced, she will be able to win the game and these extra moves will be helpful to define Duplicator's winning strategy.

We define the x-distance of two nodes $v_{i,j}$ and $v_{k,\ell}$ as $\mathrm{dist}_x(v_{i,j}, v_{k,\ell}) = |i - k|$, that is, the x-distance is 0 if the vertices are in the same column and 1 if they are not, and the y-distance as $\mathrm{dist}_y(v_{i,j}, v_{k,\ell}) = |j - \ell|$, that is, the y-distance is the number of rows between the vertices (minus 1). Note that the y-distance is not the distance in the graphs, e.g. $\mathrm{dist}_y(g_{-3}, g_{-1}) = 2^{q+1} - 1$, even though g_{-3} and g_{-1} are adjacent in G_q.

Assume now that the first i moves have been made in the game and the players have selected the vertices $\bar{g} = (g_{-3}, \ldots, g_0, g_1, \ldots, g_i)$ in G_q (where $g_1, \ldots, g_i$ were freely chosen by the players), and $\bar{h} = (h_{-3}, \ldots, h_0, h_1, \ldots, h_i)$ in H_q (where $h_1, \ldots, h_i$ were freely chosen by the players). We prove by induction that Duplicator can play in such a way that after round i of the $(\mathrm{conn}_{k,q})$-game the following conditions hold for all $-3 \leq j, \ell \leq i$:

(1) if $g_j = v_{x,y}$, then $h_j = v'_{x',y}$ – that is, corresponding pebbles are in the same row, and in particular $\mathrm{dist}_y(g_j, g_\ell) = \mathrm{dist}_y(h_j, h_\ell)$, and
(2) if $\mathrm{dist}_y(g_j, g_\ell) \leq 2^{q-i}$, then $\mathrm{dist}_x(g_j, g_\ell) = \mathrm{dist}_x(h_j, h_\ell)$.

Before showing how Duplicator can maintain this invariant, we show that these conditions together with the first four extra moves imply that the mapping $\bar{g} \mapsto \bar{h}$ is a partial isomorphism of G_q and H_q. The proof is similar to the standard proof that distances larger than 2^q in graphs cannot be expressed by first-order formulas with q quantifiers. Intuitively, the twist (the difference in indices between g_{-1}, g_0 and h_{-1}, h_0) cannot be detected in a local neighborhood.

Let us show that also for every $0 \leq \ell \leq k$ and every sequence $(i_1, \ldots, i_{\ell+2})$ of numbers in $\{-3, \ldots, i\}$ we have $G_q \models \mathrm{conn}_\ell(g_{i_1}, \ldots, g_{i_{\ell+2}})$ if and only if $H_q \models \mathrm{conn}_\ell(h_{i_1}, \ldots, h_{i_{\ell+2}})$. Assume $G_q \models \mathrm{conn}_\ell(g_{i_1}, \ldots, g_{i_{\ell+2}})$, that is, g_{i_1} and g_{i_2} are connected after the deletion of $g_{i_3}, \ldots, g_{i_{\ell+2}}$, say by a path $P = v_{x_1,y_1} \ldots v_{x_m,y_m}$, where $v_{x_1,y_1} = g_{i_1}$ and $v_{x_m,y_m} = g_{i_2}$. Then there are no $g_{i_{j_1}} = v_{x,y}$ and $g_{i_{j_2}} = v_{x',y'}$ (for $j_1, j_2 \geq 3$) with $y = y' = y_i$ and $x \neq x'$ for some $2 \leq i \leq m - 1$ (this would block a row along which the path goes, which is not possible) and no $g_{i_{j_1}} = v_{x,y}$ and $g_{i_{j_2}} = v_{x',y'}$ (for $j_1, j_2 \geq 3$) with $y_i = y = y' - 1 = y_{i+1} - 1$ and $x \neq x'$ for some $2 \leq i \leq m - 1$ (this would block a "diagonal" of which the path contains at least one vertex, which is not possible). By the first condition of the invariant there are no $h_{i_{j_1}} = v_{x,y}$ and $h_{i_{j_2}} = v_{x',y'}$ (for $j_1, j_2 \geq 3$) with $y = y' = y_i$ and $x \neq x'$ for some $2 \leq i \leq m - 1$ and by the second condition of the invariant there are no $h_{i_{j_1}} = v_{x,y}$ and $h_{i_{j_2}} = v_{x',y'}$ (for $j_1, j_2 \geq 3$) with $y_i = y = y' - 1 = y_{i+1} - 1$ and $x \neq x'$ for some $2 \leq i \leq m-1$. Now, if $P' = v'_{x_1,y_1} \ldots v'_{x_m,y_m}$ is not a path from h_{i_1} to h_{i_2} after the deletion of $h_{i_3}, \ldots, h_{i_{\ell+2}}$, it is possible to reroute the path by switching the row appropriately, as the h_{i_j} never block a complete row or a diagonal, as shown above. The case $H_q \models \mathrm{conn}_\ell(h_{i_1}, \ldots, h_{i_{\ell+2}})$ is symmetrical.

We now show that Duplicator can maintain the invariants (1) and (2) throughout the game. For the initial configuration $i = 0$, the conditions are obviously fulfilled for $-3 \leq j, \ell \leq 0$. Corresponding pebbles are in the same row and note that $\mathrm{dist}_y(g_j, g_\ell) = 2^{q+1} - 1$, for $j \in \{-3, -2\}$ and $\ell \in \{-1, 0\}$ and analogously for h_j and h_ℓ.

For the induction step, suppose that the conditions are fulfilled so far and that Spoiler is making his $(i+1)$-move in G_q (the case of H_q is symmetrical). We may assume that Spoiler does not choose a vertex that was chosen before, say Spoiler picks $g_{i+1} = v_{\_,a}$. In order to fulfill the conditions on the partial isomorphism, Duplicator must choose $h_{i+1} = v'_{\_,a}$ with the same y-coordinate. We have to make sure that she can choose the vertex with that y-coordinate satisfying the second condition. Let $g_j = v_{\_,b}$ and $g_\ell = v_{\_,c}$ with $-3 \leq j, \ell \leq i$ be such that $b \leq a \leq c$ and there is no other $g_k = v_{\_,d}$ with $b < d < c$. Intuitively, g_j is a lowest pebble that was placed above (or in the same row as) g_{i+1}, while g_k is a highest pebble that was placed below (or in the same row as) g_{i+1}.

There are two cases:

(1) $\mathrm{dist}_y(g_j, g_\ell) \leq 2^{q-i}$: Then by hypothesis, $\mathrm{dist}_x(h_j, h_\ell) = \mathrm{dist}_x(g_j, g_\ell)$ and $\mathrm{dist}_y(h_j, h_\ell) = \mathrm{dist}_y(g_j, g_\ell)$. Here, Duplicator chooses the unique $h_{i+1} = v'_{\_,a}$ such that $\mathrm{dist}_x(h_j, h_{i+1}) = \mathrm{dist}_x(g_j, g_{i+1})$, and we have $\mathrm{dist}_x(h_\ell, h_{i+1}) = \mathrm{dist}_x(g_\ell, g_{i+1})$.

(2) $\mathrm{dist}_y(g_j, g_\ell) > 2^{q-i}$: Then $\mathrm{dist}_y(h_j, h_\ell) > 2^{q-i}$ and there are three possibilities:
   - $\mathrm{dist}_y(g_j, g_{i+1}) \leq 2^{q-(i+1)}$: Then $\mathrm{dist}_y(g_\ell, g_{i+1}) > 2^{q-(i+1)}$, and Duplicator chooses $h_{i+1} = v'_{\_,a}$ such that $\mathrm{dist}_x(h_j, h_{i+1}) = \mathrm{dist}_x(g_j, g_{i+1})$. Hence, $\mathrm{dist}_y(h_\ell, h_{i+1}) > 2^{q-(i+1)}$.
   - $\mathrm{dist}_y(g_\ell, g_{i+1}) \leq 2^{q-(i+1)}$: Then $\mathrm{dist}_y(g_j, g_{i+1}) > 2^{q-(i+1)}$. Similarly to the previous case, Duplicator chooses $h_{i+1} = v'_{\_,a}$ such that $\mathrm{dist}_x(h_\ell, h_{i+1}) = \mathrm{dist}_x(g_\ell, g_{i+1})$. Consequently, $\mathrm{dist}_y(h_j, h_{i+1}) > 2^{q-(i+1)}$.
   - $\mathrm{dist}_y(g_j, g_{i+1}) > 2^{q-(i+1)}$ and $\mathrm{dist}_y(g_\ell, g_{i+1}) > 2^{q-(i+1)}$: Here, Duplicator can choose $h_{i+1} = v'_{1,a}$ or $h_{i+1} = v'_{2,a}$ as she wants. We get that $\mathrm{dist}_y(h_j, h_{i+1}) \geq 2^{q-(i+1)}$ and $\mathrm{dist}_y(h_\ell, h_{i+1}) \geq 2^{q-(i+1)}$.

Thus, in all cases, the conditions are fulfilled, and Duplicator wins the $(\mathrm{conn}_{k,q})$-game on G_q and H_q. Hence, planarity is not definable in FO + conn. □

As a graph is planar if and only if it excludes K_5 and $K_{3,3}$ as (topological) minors, we conclude that FO + conn cannot express containment of minors or topological minors. This motivates the definition of the stronger logic FO + DP in the next section, which can express the existence of disjoint paths. We will show that FO + DP can be used to express minor and topological minor containment in the next section. The disjoint paths problem gets as input a graph G and vertices $s_1, t_1, \ldots, s_k, t_k \in V(G)$. The question is whether there are pairwise vertex disjoint paths between s_i and t_i for $1 \leq i \leq k$.

Corollary 3.11. *The disjoint paths problem cannot be expressed in* FO + conn.

The proof of the next theorem is deferred to the next section, as it is a consequence of the fact that the even stronger logic FO + DP cannot express bipartiteness (Theorem 4.8).

Theorem 3.12. *Bipartiteness cannot be expressed in* FO + conn.

Finally, we show that the FO + conn_k hierarchy is strict by proving that $(k+2)$-connectivity cannot be expressed by FO + conn_k. On the other hand, $(k+2)$-connectivity can be expressed by FO + conn_{k+1} (Example 3.2).

Theorem 3.13. *$(k+2)$-connectivity cannot be expressed by* FO + conn_k. *In particular, the* FO + conn_k *hierarchy is strict, that is,* FO + $\mathrm{conn}_0 \subsetneq$ FO + $\mathrm{conn}_1 \subsetneq \ldots$

Proof. Let k be an integer. For every integer q, we choose two graphs G_q and H_q such that:

- G_q is connected,
- H_q is not connected, and
- $G_q \simeq_q H_q$.

This is possible, as connectivity is not first-order definable, and $\simeq_q$ has only finitely many equivalence classes (as there are only finitely many FO$[q]$-sentences over the signature of graphs). For example, we can choose G_q as a cycle of length 2^{q+1} and H_q as the union of two disjoint cycles of length 2^q, see e.g. the example for Theorem 4.12 of [30].

Then, we define the graph G_q^k (resp. H_q^k) as the disjoint union of G_q (resp. H_q) and K_{k+1}, a clique of size $k+1$, and connect the vertices of the clique with all vertices of G_q (resp. H_q), that is, we add the additional edges such that $(x, y) \in E(G_q^k)$ (resp. $(x, y) \in E(H_q^k)$) if $x \in G_q$ (resp. $x \in H_q$) and $y \in K_{k+1}$. The graph G_q^k is $(k+2)$-connected (the deletion of any $k+1$ vertices cannot disconnect G_q^k), while H_q^k is not $(k+2)$-connected (the deletion of the copy of K_{k+1} disconnects H_q^k). Therefore, every $\mathrm{conn}_k(x, y, z_1, \ldots, z_k)$ predicate can be expressed by an atomic plain first-order formula: in both graphs (the valuations of) x and y are not connected after the deletion of (the valuations of) $z_1, \ldots, z_k$ if and only if x or y is equal to one of the z_i. Hence, to prove $G_q^k \simeq_{\mathrm{conn}_{k,q}} H_q^k$ it suffices to prove $G_q^k \simeq_q H_q^k$ to finish the proof.

Claim 3.14. *For all integers q, k we have $G_q^k \simeq_q H_q^k$.*

Proof. The following is obviously a winning strategy for Duplicator in the q-round EF game on G_q^k and H_q^k. If Spoiler plays a pebble in the subgraph G_q or H_q, Duplicator can respond by a pebble in the subgraph H_q or G_q according to the winning strategy of Duplicator in the EF game on G_q and H_q. Otherwise, if Spoiler picks a pebble in the subgraph K_{k+1} of G_q^k or H_q^k, Duplicator can respond by a pebble in the subgraph K_{k+1} of the other graph H_q^k or G_q^k. □

This concludes the proof of Theorem 3.13. □

4 DISJOINT-PATHS LOGIC

In this section, we study the expressive power of disjoint-paths logic FO + DP. We again fix a signature σ that does not contain the symbol disjoint-paths_k for any $k \geq 1$ and that does contain a binary (edge) relation symbol E. The disjoint paths predicates will always refer to this relation. We let $\sigma\text{+disjoint-paths} := \sigma \cup \{\text{disjoint-paths}_k : k \geq 1\}$, where each symbol disjoint-paths_k is a $2k$-ary relation symbol.

Definition 4.1. The formulas of $(\mathrm{FO} + \mathrm{DP})_\sigma$ are the formulas of $\mathrm{FO}_{\sigma\text{+disjoint-paths}}$. We usually simply write FO + DP, when σ is understood from the context.

For a σ-structure $\mathfrak{A}$, an assignment $\bar{a}$ and an FO + DP formula $\varphi(\bar{x})$, we define the satisfaction relation $(\mathfrak{A}, \bar{a}) \models \varphi(\bar{x})$ as for first-order logic, where an atomic predicate $\text{disjoint-paths}_k[(x_1, y_1), \ldots (x_k, y_k)]$ is evaluated as follows. Assume that the universe of $\mathfrak{A}$ is A and let $G = (A, E^{\mathfrak{A}})$ be the graph on vertex set A and edge set $E^{\mathfrak{A}}$. Then $(\mathfrak{A}, \bar{a})$ models $\text{disjoint-paths}_k[(x_1, y_1), \ldots, (x_k, y_k)]$ if and only if in G there exist k internally vertex-disjoint paths $P_1, \ldots, P_k$, where P_i connects $\bar{a}(x_i)$ and $\bar{a}(y_i)$.

We write $\mathrm{FO} + \mathrm{DP}_k$ for the fragment of FO + DP that uses only $\text{disjoint-paths}_\ell$ predicates for $\ell \leq k$. The quantifier rank of an FO + DP formula is defined as for plain first-order logic. For structures $\mathfrak{A}$ with universe A and $\bar{a} \in A^m$ and $\mathfrak{B}$ with universe B and $\bar{b} \in B^m$, we write $(\mathfrak{A}, \bar{a}) \equiv_{\mathrm{DP}} (\mathfrak{B}, \bar{b})$ if $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{B}, \bar{b})$ satisfy the same FO + DP formulas, that is, for all $\varphi(\bar{x}) \in \mathrm{FO} + \mathrm{DP}$ we have $\mathfrak{A} \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{b})$. Similarly, we write $(\mathfrak{A}, \bar{a}) \equiv_{\mathrm{DP}_k} (\mathfrak{B}, \bar{b})$ and $(\mathfrak{A}, \bar{a}) \equiv_{\mathrm{DP}_{k,q}} (\mathfrak{B}, \bar{b})$ if $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{B}, \bar{b})$ satisfy the same $\mathrm{FO} + \mathrm{DP}_k$ formulas and the same $\mathrm{FO} + \mathrm{DP}_k$ formulas of quantifier rank at most q, respectively.

4.1 Expressive power of disjoint-paths logic

We now study the expressive power of disjoint-paths logic.

Observation 4.2. FO + conn ⊆ FO + DP because $\mathrm{conn}_k(x, y, z_1, \ldots, z_k)$ is equivalent to

$$\text{disjoint-paths}_{k+1}[(x, y), (z_1, z_1), \ldots, (z_k, z_k)] \wedge \bigwedge_{i \leq k} (z_i \neq x \wedge z_i \neq y).$$

Moreover, the inclusion is strict because planarity (Theorem 3.10) and hence, the disjoint paths problem (Corollary 3.11) is not expressible in FO + conn, while planarity and in fact the problem that a graph contains a fixed (topological) minor can be expressed in FO + DP.

Example 4.3. For every fixed graph H, there is an FO + DP formula φ_H^{top} such that $G \models \varphi_H^{top}$ if and only if $H \preccurlyeq^{top} G$.

Let n, m, ℓ respectively be the number of vertices, edges, and isolated vertices in H. Let $x_1, \dots x_n$ be n variables. Let $e_1, \dots, e_m$ be the list of edges of H, and let v_{j_s} and v_{j_t} be the two endpoints of e_j. Finally, let $v_{i_1}, \dots, v_{i_\ell}$ be the isolated vertices of H. Then,

$$\varphi_H^{top} := \exists x_1, \dots x_n \Big(\bigwedge_{i \neq j} x_i \neq x_j \ \wedge \ \text{disjoint-paths}_{m+\ell}[(x_{e_{1_s}}, x_{e_{1_t}}), \dots (x_{e_{m_s}}, x_{e_{m_t}}), (x_{i_1}, x_{i_1}), \dots (x_{i_\ell}, x_{i_\ell})]\Big).$$

Example 4.4. For every fixed graph H, there is an FO + DP formula φ_H such that $G \models \varphi_H$ if and only if $H \preccurlyeq G$. This is because, for every graph H, there exists a finite family of graphs $H_1, \dots, H_\ell$ such that $H \preccurlyeq G$ if and only if there is an $i \leq \ell$ such that $H_i \preccurlyeq^{top} G$. This family can be obtained by considering all possibilities of replacing every branch set representing a vertex of H of degree $d \geq 3$ with a tree with at most d leaves and hardcoding their shapes by disjoint paths.

Example 4.5. Planarity can be expressed in FO + DP. This is a corollary of the previous example, using the formula $\varphi_{planar} := \neg\varphi_{K_5} \wedge \neg\varphi_{K_{3,3}}$.

Example 4.6. A graph has treewidth 1 if it is a tree or a forest, hence an acyclic graph. We can express acyclicity in FO + conn (see Example 3.3).

A graph has treewidth 2 if every biconnected component is series-parallel. Series-parallel graphs exclude K_4 as a minor: $\varphi_{tw2} := \neg\varphi_{K_4}$. In general, we can express treewidth at most k, $k \in \mathbb{N}$, because it can be defined by a finite set of forbidden minors [34].

4.2 The limits of disjoint-paths logic

We now study the limits of disjoint-paths logic and show that bipartiteness cannot be expressed in FO + DP. We also show that the hierarchy on $(\text{FO} + \text{DP}_k)_{k \geq 1}$ is strict. These results are based again on an adaptation of the standard Ehrenfeucht-Fraïssé game.

The *($\text{DP}_{k,q}$)-game* is played just as the q-round EF game, but the winning condition is adapted as follows. If in q rounds the players have chosen $\bar{a} = a_1, \dots, a_q$ and $\bar{b} = b_1, \dots, b_q$, then Duplicator wins if

(1) the mapping $\bar{a} \mapsto \bar{b}$ is a partial isomorphism of $\mathfrak{A}$ and $\mathfrak{B}$, and
(2) for every $\ell \leq k$ and every sequence $(i_1, \dots, i_{2\ell})$ of numbers in $\{1, \dots, q\}$ we have

$$\mathfrak{A} \models \text{disjoint-paths}_\ell[(a_{i_1}, a_{i_2}), \dots, (a_{i_{2\ell-1}}, a_{i_{2\ell}})] \quad \iff \quad \mathfrak{B} \models \text{disjoint-paths}_\ell[(b_{i_1}, b_{i_2}), \dots, (b_{i_{2\ell-1}}, b_{i_{2\ell}})].$$

Otherwise, Spoiler wins. We say that Duplicator *wins the ($\text{DP}_{k,q}$)-game* on $\mathfrak{A}$ and $\mathfrak{B}$ if she can force a win no matter how Spoiler plays. We then write $\mathfrak{A} \simeq_{\text{DP}_{k,q}} \mathfrak{B}$.

As $(\text{FO} + \text{DP})_\sigma$ is defined as $\text{FO}_{\sigma+\text{disjoint-paths}}$ we obtain the following theorem.

Theorem 4.7. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two σ-structures where σ is purely rational (and contains a binary relation symbol E that is interpreted on both structures as an irreflexive and symmetric relation). Then $\mathfrak{A} \equiv_{\text{DP}_{k,q}} \mathfrak{B}$ if and only if $\mathfrak{A} \simeq_{\text{DP}_{k,q}} \mathfrak{B}$.*

Theorem 4.8. *Bipartiteness is not definable in* FO + DP.

Proof. Let q be an integer, and let G be a cycle graph with 2^q vertices and H a cycle graph with $2^q + 1$ vertices. Then, G is bipartite because it has an even number of vertices, and H is not bipartite because it has an odd number of vertices. We want to show that $G \simeq_{\text{DP}_{k,q}} H$ by induction over q.

We define the distance $\text{dist}(x, y)$ of two vertices x and y as the length of the shortest path between x and y.

Let $\bar{g} = (g_1, \ldots, g_i)$ be the first i moves in G and similarly $\bar{h} = (h_1, \ldots, h_i)$ the first i moves in H. We can prove by induction that Duplicator can play in such a way that after round i of the $(\mathrm{DP}_{k,q})$-game the following conditions hold for all $j, \ell \leq i$:

(1) If $\mathrm{dist}(g_j, g_\ell) < 2^{q-i+1}$, then $\mathrm{dist}(g_j, g_\ell) = \mathrm{dist}(h_j, h_\ell)$.
(2) If $\mathrm{dist}(g_j, g_\ell) \geq 2^{q-i+1}$, then $\mathrm{dist}(h_j, h_\ell) \geq 2^{q-i+1}$.
(3) The selected vertices in G and H have the same "circular order".

By the first two conditions, the partial isomorphism $\bar{g} \mapsto \bar{h}$ can be ensured. Furthermore, the third condition implies that the second condition for Duplicator's win is also satisfied.

The base case $i = 1$ of the induction is trivial because $\mathrm{dist}(g_1, g_1) = \mathrm{dist}(h_1, h_1) = 0$.

For the induction step, suppose that $G \simeq_{\mathrm{DP}_{k,i}} H$ holds and Spoiler is making his $(i+1)$-st move in G. The case of H is equivalent.

If Spoiler picks g_j for some $j \leq i$, a vertex that has already been chosen before, Duplicator can choose h_j, and the conditions are fulfilled by the induction hypothesis. Otherwise, Spoiler picks a vertex g_{i+1} that has not been chosen before. Now we have to differentiate two cases:

(1) There is only one other vertex that has already been played, $g_j = g_1, j \leq i$. Then, we can find h_{i+1} such that $\mathrm{dist}(h_1, h_{i+1}) = \mathrm{dist}(g_1, g_{i+1})$.
(2) g_{i+1} lies on the shortest path between g_j and g_ℓ with $j, \ell \leq i$ such that there is no other $g_n, n \leq i$ that lies on this path. Then, there are two possibilities:
   - $\mathrm{dist}(g_j, g_\ell) < 2^{q-i+1}$: Then $\mathrm{dist}(h_j, h_\ell) < 2^{q-i+1}$ and we can find h_{i+1} on the shortest path between h_j and h_ℓ such that $\mathrm{dist}(h_j, h_{i+1}) = \mathrm{dist}(g_j, g_{i+1})$ and $\mathrm{dist}(h_{i+1}, h_\ell) = \mathrm{dist}(g_{i+1}, g_\ell)$.
   - $\mathrm{dist}(g_j, g_\ell) \geq 2^{q-i+1}$: Then $\mathrm{dist}(h_j, h_\ell) \geq 2^{q-i+1}$ and there are three cases:
   (a) $\mathrm{dist}(g_j, g_{i+1}) < 2^{q-i}$: Then $\mathrm{dist}(g_{i+1}, g_\ell) \geq 2^{q-i}$ and we can choose h_{i+1} on the shortest path between h_j and h_ℓ such that $\mathrm{dist}(h_j, h_{i+1}) = \mathrm{dist}(g_j, g_{i+1})$ and $\mathrm{dist}(h_{i+1}, h_\ell) \geq 2^{q-i}$.
   (b) $\mathrm{dist}(g_{i+1}, g_\ell) < 2^{q-i}$: This case is similar to the previous one.
   (c) $\mathrm{dist}(g_j, g_{i+1}) \geq 2^{q-i}$ and $\mathrm{dist}(g_{i+1}, g_\ell) \geq 2^{q-i}$: Since $\mathrm{dist}(h_j, h_\ell) \geq 2^{q-i+1}$, we can find h_{i+1} with $\mathrm{dist}(h_j, h_{i+1}) \geq 2^{q-i}$ and $\mathrm{dist}(h_{i+1}, h_\ell) \geq 2^{q-i}$ in the middle of the shortest path between h_j and h_ℓ.

Thus, in all cases, the conditions are fulfilled. This completes the inductive proof. □

We now show that the hierarchy on $(\mathrm{FO} + \mathrm{DP}_k)_{k \geq 1}$ is strict.

Lemma 4.9. *For all integers $k \geq 1$, $(2k)$-connectivity is not expressible in* $\mathrm{FO} + \mathrm{DP}_k$.

Proof. Let k be an integer. For every integer q, we define two graphs G_q and H_q such that:

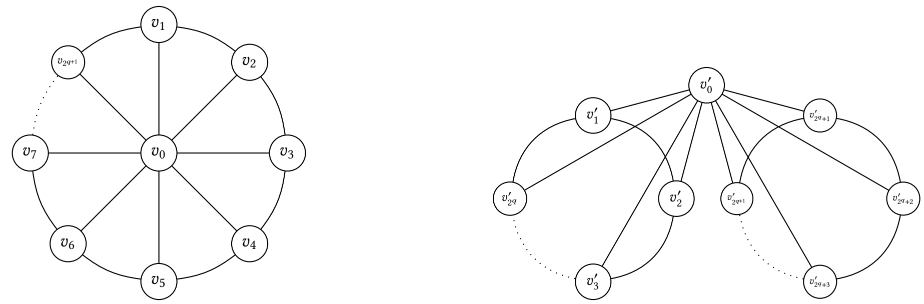
- G_q is 2-connected,
- H_q is 1-connected but not 2-connected, and
- $G_q \simeq_q H_q$

For example, take G_q the cycle with 2^{q+1} many elements, together with an apex vertex, while H_q is the disjoint union of two cycles with 2^q many elements each, together with an apex vertex (see Figure 2).

Obviously, G_q is 2-connected and H_q is 1-connected but not 2-connected. To show that G_q and H_q are $\mathrm{FO}[q]$-equivalent, we can play the EF-game in the same way as for connectivity on a cycle and a disjoint union of two cycles with the only difference that Duplicator chooses the apex vertex of the other graph whenever Spoiler chooses the apex vertex of one graph in his move.

We then define G_q^k (resp. H_q^k) as the lexicographical product of G_q (resp. H_q) with K_{2k}, the clique with $2k$ elements. More precisely, if $G_q = (V, E)$, where $V = \{1, \ldots, n\}$, then $G_q^k := (V', E')$ where:

- $V' := \{v_{1,1}, \ldots, v_{1,2k}, \ldots, v_{n,1}, \ldots, v_{n,2k}\}$



(a) G_q (b) H_q

Fig. 2. The FO + DP hierarchy is strict

- $E' := \{\{v_{i,j}, v_{i',j'}\} \ : \ i = i' \lor (i, i') \in E\}$.

One can view G_q^k as $2k$ copies of G_q on top of each other. Vertices are replaced by $2k$-cliques, and edges are replaced by $(2k, 2k)$-bicliques. A direct consequence of the definition is the following equivalence.

Claim 4.10. *For all integers q, k, we have that $G_q^k \simeq_q H_q^k$.*

Proof. Duplicator's strategy follows the one derived from $G_q \simeq_q H_q$. If Spoiler picks a vertex $v_{i,j} \in G_q^k$, then Duplicator can respond by choosing the vertex $v_{i',j} \in H_q^k$ where $v_{i'} \in H_q$ is Duplicator's response to $v_i \in G_q$. □

We then show that over G_q^k and H_q^k, the predicate disjoint-paths$_k$[ ] is always true and therefore that, for these structures, $(\mathrm{FO} + \mathrm{DP}_k)[q]$ collapses to $\mathrm{FO}[q]$.

Claim 4.11. *For all integers q, k, for every k-tuples $\bar{a}, \bar{b}$, we have that G_q^k and H_q^k both satisfy the query* disjoint-paths$_k[(a_1, b_1), \ldots, (a_k, b_k)]$.

Proof. The proofs for G_q^k and H_q^k are identical, so we only do it for G_q^k. Remember that n is the number of vertices in G_q. The idea is that each of the k paths uses at most two "copies" of each vertex of G_q, hence $2k$ "copies" is enough for all paths to exist. For every $i \leq n$, let $B_i := \{v_{i,j} \ : \ j \leq 2k\}$, and $F_i := \{v_{i,j} \ : \ j \leq 2k \land v_{i,j} \notin \bar{a} \land v_{i,j} \notin \bar{b}\}$. We call B_i the set of vertices in *position* i, and F_i the *free vertices* in position i. We then compute each path, starting with (a_1, b_1).

Let i, j, i', j' such that $a_1 = v_{i,j}$ and $b_1 = v_{i',j'}$. If $i = i'$, then there is nothing to do as a_1 and b_1 are neighbors. Otherwise, note that for every $i'' \leq n$, $F_{i''} \neq \emptyset$, because there are only $2k - 2$ elements among $a_2, \ldots, a_k, b_2, \ldots, b_k$. Since G_q is a connected graph, there is a path from i to i'. For every inner node i'' of this path, we can select a vertex $v \in F_{i''}$. We can therefore create a path in G_q^k from a_1 to b_1 where all inner vertices are free vertices. We then remove these vertices from the sets of free vertices.

Let now $1 < \ell \leq k$, and let i, j, i', j' such that $a_\ell = v_{i,j}$ and $b_\ell = v_{i',j'}$. We assume that the first $\ell - 1$ paths have already been computed. Observe that here again, if $i = i'$ there is nothing to do. Otherwise, we again have that for every i'', $F_{i''}$ is not empty. This is because for every $s \leq k$, the path from a_s to b_s intersects $B_{i''}$ at most twice *(at most once for the inner vertices, and twice when the two endpoints are both in position i'')*. Therefore, we can select a path in G_q from i to i' and for each i'' in this path, pick a vertex $v \in F_{i''}$. □

With Claim 4.11, we can replace formulas of $(\mathrm{FO} + \mathrm{DP}_k)[q]$ by formulas of $\mathrm{FO}[q]$. Thanks to Claim 4.10, $G_q^k \simeq_q H_q^k$, we conclude that $G_q^k \simeq_{\mathrm{DP}_{k,q}} H_q^k$. So FO + DP$_k$ cannot express $2k$-connectivity. Note that this bound is tight for these structures i.e. $G_q^k \not\simeq_{\mathrm{DP}_{k+1,q}} H_q^k$. □

Since $2k$-connectivity is expressible in $\mathrm{FO} + \mathrm{conn}_{2k-1}$ (see Example 3.2) but the disjoint paths problem is not expressible in FO + conn (see Corollary 3.11), we can conclude the following corollary.

Corollary 4.12. *$\mathrm{FO} + \mathrm{DP}_k$ and $\mathrm{FO} + \mathrm{conn}_{2k-1}$ are not comparable for $k \geq 2$.*

We believe that $\mathrm{FO} + \mathrm{DP}_k$ cannot express $(k+1)$-connectivity and hence, also $\mathrm{FO} + \mathrm{DP}_k$ and $\mathrm{FO} + \mathrm{conn}_k$ are not comparable for $k \geq 2$.

Conjecture 4.13. *$\mathrm{FO} + \mathrm{DP}_k$ and $\mathrm{FO} + \mathrm{conn}_k$ are not comparable for $k \geq 2$.*

Lemma 4.14. *The $\mathrm{FO} + \mathrm{DP}_k$ hierarchy is strict, that is, $\mathrm{FO} + \mathrm{DP}_1 \subsetneq \mathrm{FO} + \mathrm{DP}_2 \subsetneq \dots$*

Proof. Consider the structures in the proof of Lemma 4.9, which are indistinguishable in $\mathrm{FO} + \mathrm{DP}_k$. The following sentence of $\mathrm{FO} + \mathrm{DP}_{k+1}$ distinguishes G_q^k and H_q^k:

$$\exists a_1 \exists b_1 \dots \exists a_{k+1} \exists b_{k+1} \neg \text{disjoint-paths}_{k+1}[(a_1, b_1), \dots, (a_{k+1}, b_{k+1})]$$

In H_q^k, pick the vertex i such that the induced subgraph $H_q \setminus \{i\}$ is not connected. Let $i', i'' \in V(H_q)$ be two vertices that are not connected in $H_q \setminus \{i\}$. Then pick the vertices $a_j = v_{i,j}$ for $j \leq k$ and $b_j = v_{i,k+j}$ for $j \leq k$, as well as $a_{k+1} = v_{i',1}$ and $b_{k+1} = v_{i'',1}$.

Intuitively, this means that the path between $a_{k+1} = v_{i',1}$ and $b_{k+1} = v_{i'',1}$ needs to traverse at least one of the vertices $v_{i,\_}$ because the vertices $v_{i',\_}$ and $v_{i'',\_}$ are not connected in $H_q^k \setminus \bigcup_{j \leq 2k} \{v_{i,j}\}$. However, the path between $a_{k+1} = v_{i',1}$ and $b_{k+1} = v_{i'',1}$ also has to be internally vertex-disjoint to the paths between $a_j = v_{i,j}$ and $b_j = v_{i,k+j}$ for $j \leq k$ whose endpoints are all vertices $v_{i,\_}$. Therefore, there are no $k+1$ disjoint path between the a_j's and b_j's and the $\mathrm{FO} + \mathrm{DP}_{k+1}$-sentence $\exists a_1 \exists b_1 \dots \exists a_{k+1} \exists b_{k+1} \neg \text{disjoint-paths}_{k+1}[(a_1, b_1), \dots, (a_{k+1}, b_{k+1})]$ is satisfied in H_q^k.

G_q^k does not satisfy this $\mathrm{FO} + \mathrm{DP}_{k+1}$-formula because there is no vertex $i \in V(G_q)$ such that the induced subgraph $G_q \setminus \{i\}$ is not connected. Instead, we can find $k+1$ disjoint paths for all given pairs of vertices in G_q^k. □

4.3 Equivalent operators

It seems natural to consider other operators that can express the presence or exclusion of a minor or topological minor. Consider the following operators.

(1) $\mathrm{minor}(x_1, \dots, x_k, H)$, expressing that G contains H as a minor with branch sets $H_1, \dots, H_k \subseteq G$ such that $x_i \in V(H_i)$.
(2) $\text{top-minor}(x_1, \dots, x_k, H)$, expressing that G contains H as a topological minor with principal vertices $x_i \in V(H_i)$, where H is an ordered graph such that the vertices x_i are uniquely associated to the vertices of H.

Observation 4.15. Disjoint-paths logic, minor logic, and topological-minor logic have the same expressive power.

Proof. Disjoint-paths logic can express the presence of a minor or topological minor (Examples 4.3 and 4.4). Both minor and topological-minor logic can express disjoint paths: $\text{disjoint-paths}_k[(x_1, y_1), \dots (x_k, y_k)]$ is equivalent to $\mathrm{minor}(x_1, y_1, \dots, x_k, y_k, M_k)$, where M_k is a graph on vertex set $x_1, y_1, \dots, x_k, y_k$ inducing a matching between the x_i and y_i.

□

It will be interesting to study a variation of minor logic where we adapt the minor operator such that $\mathrm{minor}(x_1, \dots, x_k, H)$ expresses that G contains H as a minor after the deletion of $x_1, \dots, x_k$ (not specifying the vertices that must be contained in the branch sets).

A second variation is the minor$(y, x_1, \ldots, x_k, H)$ operator, expressing that after the deletion of $x_1, \ldots, x_k$, G contains H as a minor in the component of y.

5 CONNECTION TO OTHER LOGICS

In this section, we compare the expressive power of separator logic and disjoint-paths logic with monadic second-order logic and transitive-closure logic. Figure 5 depicts the connections between these logics.

5.1 Monadic second-order logic

Monadic second-order logic (MSO_1) allows quantification over sets of vertices in addition to the first-order quantifiers. It has a higher expressive power than first-order logic because, for example, connectivity is expressible in MSO_1. Connectivity is expressible by

$$\forall R\Big(\big(\exists x R(x) \wedge \exists x \neg R(x)\big) \to \exists x \exists y \big(R(x) \wedge \neg R(y) \wedge E(x,y)\big)\Big)$$

By an extension of this formula, we can say that a given set S is connected:

$$\begin{aligned}\text{conn-set}(S) := \forall R\Big(&\big(R \subseteq S \wedge \exists x\, R(x) \wedge \exists x\, (S(x) \wedge \neg R(x))\big)\\ &\to \exists x \exists y \big(R(x) \wedge \neg R(y) \wedge S(y) \wedge E(x,y)\big)\Big)\end{aligned}$$

Furthermore, we can express the connectivity operators in MSO_1. The connectivity operator $\mathrm{conn}_0(x,y)$ can be expressed by:

$$\mathrm{conn}_0(x,y) := \forall R\Big(R(x) \wedge \forall v \forall w \big((R(v) \wedge E(v,w)) \to R(w)\big) \to R(y)\Big)$$

and $\mathrm{conn}_k(x, y, z_1, \ldots, z_k)$ using conn-set(S) by:

$$\mathrm{conn}_k(x, y, z_1, \ldots, z_k) := \exists S\ \big(\text{conn-set}(S) \wedge S(x) \wedge S(y) \wedge \bigwedge_{i \le k} \neg S(z_i)\big).$$

We can express the disjoint paths predicates disjoint-paths$_k[(x_1, y_1), \ldots, (x_k, y_k)]$ by:

$$\begin{aligned}\exists S_1 \ldots \exists S_k \Bigg(&\bigwedge_{i \le k} \Big(S_i(x_i) \wedge S_i(y_i) \wedge \text{conn-set}(S_i)\Big)\\ &\wedge \bigwedge_{i<j\le k} \forall z \Big(\big(S_i(z) \wedge S_j(z)\big) \to \big((z = x_i \vee z = y_i) \wedge (z = x_j \vee z = y_j)\big)\Big)\Bigg)\end{aligned}$$

Since the disjoint paths operators are expressible in MSO_1, FO + DP is included in MSO_1. This inclusion is strict because it is well-known that bipartiteness is expressible in MSO_1:

$$\exists R_1 \exists R_2 \Big(\forall x \big(R_1(x) \leftrightarrow \neg R_2(x)\big) \wedge \bigwedge_{i \le 2} \forall x \forall y \big((R_i(x) \wedge R_i(y)) \to \neg E(x,y)\big)\Big)$$

but we showed in Theorem 4.8 that bipartiteness is not expressible in FO + DP.

5.2 Transitive-closure logic

Transitive-closure logic TC^i_j is the enrichment of first-order logic with the transitive-closure operator $[\mathrm{TC}_{\bar{x},\bar{y}}\varphi(\bar{x}, \bar{y})]$ where $\bar{x}$ and $\bar{y}$ are tuples of length i and φ is a formula with at most j free variables other than $\bar{x}$ and $\bar{y}$. The transitive-closure formula $[\mathrm{TC}_{\bar{x},\bar{y}}\varphi(\bar{x}, \bar{y})](\bar{u}, \bar{v})$ is true in a graph G if there exists tuples of vertices $\bar{z}_0, \ldots, \bar{z}_r \in V(G)^i$ with $\bar{u} = \bar{z}_0$ and $\bar{v} = \bar{z}_r$ such that $G \models \varphi(\bar{z}_\ell, \bar{z}_{\ell+1})$ for all $\ell < r$.

Every FO + conn_k formula can be expressed in TC^1_k because the conn_k operator can be expressed with the help of the transitive-closure operator:

$$\text{conn}_k(x, y, z_1, \ldots, z_k) = [\text{TC}_{v,w} E(v, w) \wedge v \neq z_1 \wedge \ldots \wedge v \neq z_k \wedge w \neq z_1 \wedge \ldots \wedge w \neq z_k](x, y)$$

In fact, TC^1_k is more expressible than FO + conn_k, as it can express bipartiteness [25, Example 7.2]. We repeat the example for readers not familiar with transitive-closure logics. A graph is bipartite if and only if it does not contain an odd cycle, which is expressed by the following formula:

$$\neg\exists u \exists v \left([\text{TC}_{x,y} \exists z (E(x, z) \wedge E(z, y))](u, v) \wedge E(v, u)\right)$$

On the other hand, 2-connectivity can naturally be expressed in FO + conn_1, but, as we prove next, not in TC^1_0. We thank Martin Grohe for pointing us to the proof idea of the following theorem.

Theorem 5.1. *2-connectivity cannot be expressed in* TC^1_0.

To prove this theorem, we construct two graphs that cannot be distinguished by transitive closure logic TC^1_0 but only one of the graphs is 2-connected. We will rely on Gaifman's Locality Theorem [21]. Let G be a graph and $r > 0$ an integer. For $v \in V(G)$ we write $N_r(v)$ for the r-neighborhood of v, that is, the set of vertices at distance at most r from v. For a tuple $\bar{v}$ of vertices we let $N_r(\bar{v}) = \bigcup_{v \in \bar{v}} N_r(v)$. A formula $\varphi(\bar{x})$ over graphs is called *r-local* if for every graph G and every $|\bar{x}|$-tuple $\bar{v}$ we have $G \models \varphi(v) \Leftrightarrow G[N_r(\bar{v})] \models \varphi(\bar{v})$. We write $\varphi^{(r)}$ to indicate that φ is r-local. Note that for every fixed r there exists an FO-formula $\text{dist}(x, y) > r$, stating that the distance between x and y is greater than r.

Theorem 5.2 (Gaifman's Locality Theorem [21] (adapted for graphs)). *Every* FO *formula* $\varphi(\bar{x})$ *over graphs is equivalent to a Boolean combination of the following:*

- *local formulas* $\psi^{(r)}(\bar{x})$ *around* $\bar{x}$*;*
- *basic local sentences (with parameters* r *and* s*) of the form*

$$\exists x_1 \ldots \exists x_s \left(\bigwedge_{i=1}^{s} \chi^{(r)}(x_i) \wedge \bigwedge_{1 \leq i < j \leq s} \text{dist}(x_i, x_j) > 2r \right)$$

for some r*-local* FO *formula* $\chi^{(r)}$.

Furthermore, if $\text{qr}(\varphi) = q$*, then* $r \leq 7^q, s \leq q + |\bar{x}|$*. If* φ *is a sentence, then only basic local sentences appear in the Boolean combination.*

In regular high-girth graphs, we have the following corollary. Recall that the girth g of a graph G is the length of a shortest cycle in G.

Corollary 5.3. *Let* $q, d > 0$ *be integers, let* $r = 7^q$*,* $s = q + 2$ *and* $g = 4r + 1$*. Let* G, H *be two* d*-regular graphs of girth at least* g *with* $|V(G)|, |V(H)| \geq s \cdot d^{2r+1}$ *and* $\varphi \in \text{FO}[q]$*. Then*

(1) G *and* H *satisfy the same basic local sentences with parameters* r *and* s*. As a consequence,* $G \equiv_q H$*.*
(2) *If* $u, v \in V(G)$ *and* $u', v' \in V(H)$ *with* $\text{dist}(u, v) > 2r$ *and* $\text{dist}(u', v') > 2r$*, then* $G \models \varphi(u, v) \Leftrightarrow H \models \varphi(u', v')$*.*
(3) *If* $u, v \in V(G)$ *and* $u', v' \in V(H)$ *with* $\text{dist}(u, v) = \text{dist}(u', v')$*, then* $G \models \varphi(u, v) \Leftrightarrow H \models \varphi(u', v')$*.*

Proof. As G and H are d-regular and have girth at least $g = 4r + 1$, for all $u, v \in V(G)$, $u', v' \in V(H)$ we have $G[N_r(u)] \cong G[N_r(v)] \cong H[N_r(u')] \cong H[N_r(v')]$; that is, the r-neighborhoods of all vertices are isomorphic (the neighborhoods induce d-regular trees). Consequently, for every r-local formula $\chi(x)$ and all $u, v \in V(G)$ and $u', v' \in V(H)$ we have $G \models \chi(u) \Leftrightarrow G \models \chi(v) \Leftrightarrow H \models \chi(u') \Leftrightarrow H \models \chi(v')$. As G is d-regular, we have $|N_{2r}(v)| \leq d^{2r+1}$ for all $v \in V(G) \cup V(H)$. As $|V(G)|, |V(H)| \geq s \cdot d^{2r+1}$, if for some (and hence for all) $v \in V(G)$ we have $G \models \chi(v)$ and hence for some (and hence for all) $v' \in V(H)$ we have $H \models \chi(v')$, then there exist at least

s vertices in G and in H that satisfy χ and have pairwise distance greater than $2r$ (iteratively choose vertices and remove the $2r$-neighborhood, so that the next vertex can be chosen without conflicts). Hence, G and H satisfy the same basic local sentences with parameters r and s. As χ was chosen as an arbitrary r-local formula, by Theorem 5.2 we have $G \equiv_q H$.

For the second statement, translate $\varphi(x, y)$ into Gaifman normal form by using Theorem 5.2. By the first statement, G and H satisfy the same basic local sentences with parameters r and s. Hence, we need to prove only that all $u, v \in V(G)$ and $u', v' \in V(G)$ with $\text{dist}(u, v) > 2r$ and $\text{dist}(u', v') > 2r$ satisfy the same r-local formulas. This however is clear, as $N_r(u, v) \cong N_r(u', v')$ with isomorphisms that map u to u' and v to v' (the neighborhoods induce forests consisting of two d-regular trees).

Similarly, for the third statement, if $\text{dist}(u, v) = \text{dist}(u', v') \le 2r$, then $G[N_r(u, v)] \cong H[N_r(u', v')]$ (by the assumption on the girth of G and H, these neighborhoods induce isomorphic d-regular trees with two distinguished vertices at the same distance) with isomorphisms that map u to u' and v to v'. Hence, $G \models \varphi(u, v) \Leftrightarrow H \models \varphi(u', v')$. ◻

We now construct two 12-regular graphs of high girth, G_q and H_q, where H_q is 2-connected, but G_q is only 1-connected and not 2-connected. Our construction is based on Cayley graphs, which encode the abstract structure of groups. We do not care about the concrete constructions of Cayley graphs but only about their nice properties.

Lemma 5.4. *For every $q \in \mathbb{N}$ there exists a graph C_q that is 12-regular, 2-connected, has girth $g = 7^q! + 1$, and a unique cycle of length g (all other cycles are longer).*

Proof. It is known that every finite connected Cayley graph of degree d is $\left\lceil \frac{2(d+1)}{3} \right\rceil$-connected [2, Theorem 3.7] and there exist arbitrarily large (connected) d-regular Cayley graphs C whose girth is $g' \ge \log_{d-1} |C|$ [14]. Therefore, there exists a 12-regular Cayley graph C that is 9-connected and has girth $g' \ge \log_{11} |C|$ where the girth only depends on the size of the graph.

We take such a Cayley graph C'_q with a girth $g' > 2 \cdot 7^q! + 2$. Then, there exists a cycle of length g' in C'_q. We now choose four vertices $v_1, \ldots, v_4$ of this cycle such that $\text{dist}(v_1, v_3) = 1, \text{dist}(v_2, v_4) = 1$ and $\text{dist}(v_1, v_2) = 7^q!$. By removing the edges (v_1, v_3) and (v_2, v_4) and adding the new edges (v_1, v_2) and (v_3, v_4) (see Figure 3), we obtain the graph C_q.

By construction, C'_q is 12-regular and 9-connected. Every vertex in C_q still has twelve neighbors and even by removing the edges (v_1, v_3) and (v_2, v_4), the graph stays at least 2-connected.

Concerning the girth, we constructed a unique cycle of length g going from v_1 over v_2 to v_1. Furthermore, there are no shorter cycles: Every cycle that does not use the edge $\{v_1, v_2\}$ has length greater than g'. ◻

We can now use this graph C_q to construct the graphs G_q and H_q where H_q is 2-connected, but G_q is not 2-connected, see Figure 4. To this end, we take six disjoint copies of the graph C_q, namely $A_1, \ldots, A_6$, and connect them in the following way: To construct the graph G_q, we choose in the six components $A_1, \ldots, A_6$ two adjacent vertices $a_i, a'_i \in V(A_i)$ for every $1 \le i \le 6$ that do not lie on the unique cycle of length g. Then, we remove the edges $\{a_i, a'_i\}$ for every $1 \le i \le 6$ and add a new vertex a and new edges $\{a, a_i\}, \{a, a'_i\}$ for every $1 \le i \le 6$. The graph G_q is thus defined as:

$$\begin{aligned} G_q = (&V(A_1) \cup \ldots \cup V(A_6) \cup \{a\}, \\ &(E(A_1) \setminus \{\{a_1, a'_1\}\}) \cup \ldots \cup (E(A_6) \setminus \{\{a_6, a'_6\}\}) \cup \{\{a, a_1\}, \{a, a'_1\}, \ldots, \{a, a_6\}, \{a, a'_6\}\}) \end{aligned}$$

To construct the graph H_q, we take the same steps as for G_q but twice: We also take six disjoint copies of C_q, namely $A_1, \ldots, A_6$. Then, we choose in the six components $A_1, \ldots, A_6$ two adjacent vertices $a_i, a'_i \in V(A_i)$ for every $1 \le i \le 6$ that do not lie in the constructed cycle of length g. Additionally, we choose two adjacent vertices $b_i, b'_i \in V(A_i)$ for every $1 \le i \le 6$ that do not lie on the constructed cycle of length g such that the vertices a_i

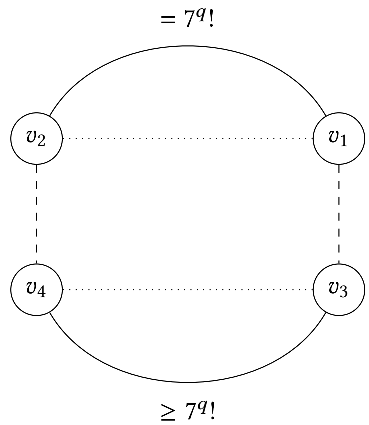
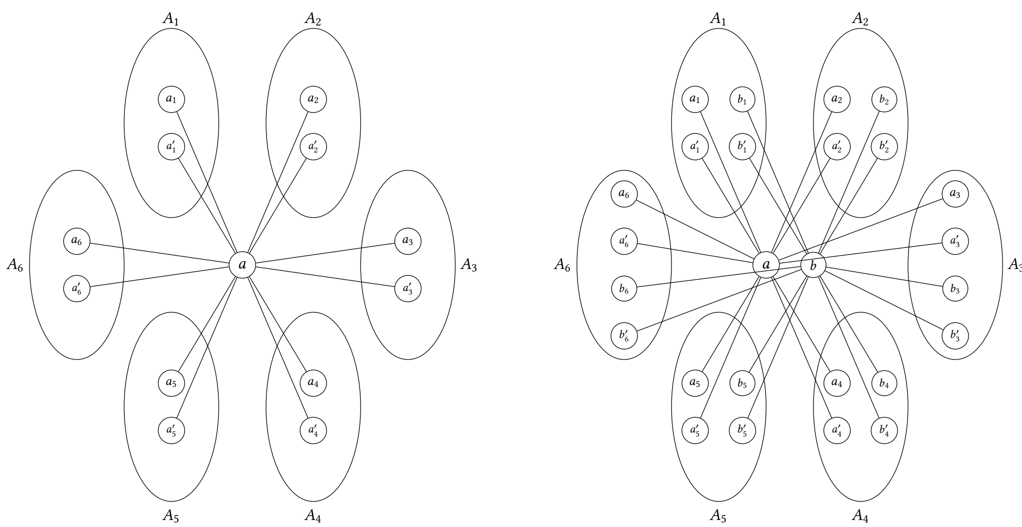


Fig. 3. The cycle of length $g' > 2 \cdot 7^q! + 2$. By removing the dashed edges and adding the dotted edges, we obtain a unique cycle of length $g = 7^q! + 1$.



(a) G_q (b) H_q

Fig. 4. Construction of G_q and H_q

and b_i as well as the vertices a_i' and b_i' have distance at least g. Then, we remove the edges $\{a_i, a_i'\}, \{b_i, b_i'\}$ for every $1 \leq i \leq 6$ and add two new vertices a and b and new edges $\{a, a_i\}, \{a, a_i'\}, \{b, b_i\}, \{b, b_i'\}$ for every $1 \leq i \leq 6$.

Formally, the graph H_q is defined as:

$$\begin{aligned} H_q = (&V(A_1) \cup \ldots \cup V(A_6) \cup \{a, b\}, \\ &(E(A_1) \setminus \{\{a_1, a_1'\}, \{b_1, b_1'\}\}) \cup \ldots \cup (E(A_6) \setminus \{\{a_6, a_6'\}, \{b_6, b_6'\}\}) \\ &\cup \{\{a, a_1\}, \{a, a_1'\}, \{b, b_1\}, \{b, b_1'\}, \ldots, \{a, a_6\}, \{a, a_6'\}, \{b, b_6\}, \{b, b_6'\}\}) \end{aligned}$$

Lemma 5.5. *The constructed graphs G_q and H_q are 12-regular and have girth $g = 7^q! + 1$. Furthermore, H_q is 2-connected and G_q is connected but not 2-connected.*

Proof. By construction, the graphs G_q and H_q are 12-regular because the components $A_1, \ldots, A_6$ are 12-regular and the added vertices a and b also have twelve neighbors. Furthermore, they have girth g because the components $A_1, \ldots, A_6$ have girth g, and we do not destroy the cycles of length g or introduce shorter cycles.

Both graphs, G_q and H_q are connected. However, G_q is not 2-connected because it becomes disconnected by removing the vertex $a \in V(G_q)$. The graph H_q only becomes disconnected by removing the vertices a and b or more vertices because $A_1, \ldots, A_6$ are 2-connected as well. Hence, H_q is 2-connected. □

In what follows, we show that (for sufficiently large q) the graphs G_q and H_q cannot be distinguished by TC_0^1 formulas. More precisely, we show that over these graphs, the TC_0^1 operator is useless: Either every pair of vertices is a solution, or none is. To show this, we first prove that between any two vertices, there is an r-walk.

Definition 5.6. Given two vertices a, b in a graph G and an integer r, an *r-walk* from a to b is a sequence $c_0, c_1, \ldots c_m$ such that $c_0 = a, c_m = b$ and $\forall i < m : \mathrm{dist}(c_i, c_{i+1}) = r$.

Note that the existence of an r-walk between a and b does not imply that the distance of a and b is a multiple of r as the r-walk might not go through the shortest path. Note also that the existence of a walk from a to b of length a multiple of r might not imply the existence of an r-walk. For example in a graph with two adjacent vertices a, b, there is a walk $a - b - a - b$ of length 3, while there is no 3-walk from a to b.

Lemma 5.7. *For all integers q, r with $r \leq 7^q$ and for all $a, b \in V(G_q)$ (resp. for all $a, b \in V(H_q)$) there exists an r-walk from a to b.*

Proof. Let $g = 7^q! + 1$ be the girth of G_q (resp. H_q). Recall that the shortest cycle S of length g is unique by construction. Since $r \leq 7^q$ divides $7^q!$, it follows that $g \equiv 1[r]$.

Let a' be a vertex of S such that $\mathrm{dist}(a, a') \equiv 0[r]$ and b' be a vertex of S such that $\mathrm{dist}(b, b') \equiv 0[r]$. This implies that there is an r-walk from a to a' and from b to b'.

Let $d = \mathrm{dist}(a', b')$. We can define an r-walk from a' to b' that traverses the cycle S m many times where we choose m such that $m - d \equiv 0[r]$. Such m exists since the length of S is $1[r]$. Observe now that we can concatenate the r-walks from a to a', from a' to b' and from b' to b. □

Lemma 5.8. *For all TC_0^1 formulas $\varphi = [\mathrm{TC}_{u,v}\Psi(u, v)](x, y)$ there exists $q \in \mathbb{N}$ such that either*

$$\begin{array}{lll} (1) & G_q \models \forall x, y\ \varphi(x, y) & \textit{and} \quad H_q \models \forall x, y\ \varphi(x, y), \textit{or} \\ (2) & G_q \models \forall x, y\ \neg\varphi(x, y) & \textit{and} \quad H_q \models \forall x, y\ \neg\varphi(x, y). \end{array}$$

In a nutshell, Lemma 5.8 shows that the graphs G_q and H_q are too regular for TC operators to define anything else than true or false statements. The proof is performed by induction on the nesting of the TC operator. In the base case, Ψ is an FO formula.

Proof. Let $\varphi = [\mathrm{TC}_{u,v}\Psi(u, v)](x, y)$ and let q be the quantifier rank of Ψ, that is $\Psi \in \mathrm{FO}[q]$. Let $r_0 = 7^q$.

If there are no elements u, v in $V(G_q)$ nor in $V(H_q)$ satisfying φ, then (2) of Lemma 5.8 holds. Assume now that there exist $a, b \in V(G_q)$ such that $G_q \models \varphi(a, b)$. By the definition of the TC operator, there exists a sequence $c_0, \ldots, c_m$ with $c_0 = a$ and $c_m = b$ such that $G_q \models \Psi(c_i, c_{i+1})$ for all $i < m$. Let $r_1 = \mathrm{dist}(c_0, c_1)$.

(1) Assume first $r_1 \leq r_0$. By Lemma 5.7, for all $u, v \in V(G_q)$ (resp. in $V(H_q)$) there exists an r_1-walk $d_0, \ldots, d_n$ with $d_0 = u$ and $d_n = v$ and G_q (resp. H_q) is a model of $\Psi(d_i, d_{i+1})$ for every $i < n$. By the definition of the TC operator, G_q (resp. H_q) is a model of $\varphi(u, v)$.

(2) If $r_1 > r_0$, then for all u, v in G_q (resp. H_q) there exists w such that $\mathrm{dist}(u, w) > r_0$ and $\mathrm{dist}(w, v) > r_0$ (since G_q is connected and contains a cycle of length $> 2r_0 + 1$) and then by Corollary 5.3 we have that G_q (resp. H_q) is a model of $\Psi(u, w)$ and $\Psi(w, v)$. By the definition of the TC operator, G_q (resp. H_q) is a model of $\varphi(u, v)$.

The case where the elements a, b satisfying φ are found in H_q is analogous.

For TC formulas $\varphi = [\mathrm{TC}_{u,v}\Psi(u, v)](x, y)$ where Ψ is not in FO, we can apply this procedure, replacing inductively the uses of TC operator by either the True or the False predicates. □

We can then conclude that every TC^1_0 formula (of quantifier rank at most q) is equivalent to an FO formula on the graphs G_q and H_q. This implies that no TC^1_0 formula expresses 2-connectivity. This concludes the proof of Theorem 5.1. We conjecture that the statement of Theorem 5.1 generalizes to higher values of k. However, using the same proof idea, the proof for general $k \in \mathbb{N}$ would be more technical. The construction of the graphs might be similar but we would need to handle additional free variables in the transitive-closure operator which would result in more difficult proofs to show that both graphs model the same TC^1_k-formulas.

Conjecture 5.9. *For every integer k, $(k + 2)$-connectivity cannot be expressed in TC^1_k.*

$$
\begin{array}{ccccccccccc}
 & & \mathrm{FO}+\mathrm{DP}_1 & \subsetneq & \mathrm{FO}+\mathrm{DP}_2 & \subsetneq & \cdots & \subsetneq & \mathrm{FO}+\mathrm{DP}_{k+1} & & \\
 & & \rotatebox{90}{$\equiv$} & & \rotatebox{90}{$\subsetneq$} & & & & \rotatebox{90}{$\subsetneq$} & \rotatebox{-45}{$\subsetneq$} & \\
\mathrm{FO} & \subsetneq & \mathrm{FO}+\mathrm{conn}_0 & \subsetneq & \mathrm{FO}+\mathrm{conn}_1 & \subsetneq & \cdots & \subsetneq & \mathrm{FO}+\mathrm{conn}_k & \subsetneq & \mathrm{MSO} \\
 & & \rotatebox{90}{$\supsetneq$} & \not\subseteq\ \not\supseteq & \rotatebox{90}{$\supsetneq$} & & & & \rotatebox{90}{$\supsetneq$} & \rotatebox{45}{$\supsetneq$} & \\
 & & \mathrm{TC}^1_0 & \subseteq & \mathrm{TC}^1_1 & \subseteq & \cdots & \subseteq & \mathrm{TC}^1_k & &
\end{array}
$$

Fig. 5. Connections between the logics

6 CONCLUSION

We studied first-order logic enriched with connectivity predicates tailored to express algorithmic graph problems that are commonly studied in contemporary parameterized algorithmics. This yields separator logic, which can query connectivity after the deletion of a bounded number of elements, and disjoint-paths logic, which can express the disjoint paths problem. We demonstrated a rich expressiveness that arises from the interplay of these predicates with the nested quantification of first-order logic. We also studied the limits of expressiveness of these new logics.

In a companion paper, we studied the model checking problem for separator logic and proved that it is fixed-parameter tractable parameterized by formula size on classes of graphs that exclude a fixed topological minor [32]. This yields a powerful algorithmic meta-theorem for separator logic.

Using the same methods it is easy to show that model checking for formulas using only conn_1 predicates is fixed-parameter tractable on nowhere dense classes of graphs, which are even more general than classes excluding a topological minor. To obtain this latter result, observe that the block decomposition of a graph can be understood as a tree decomposition with adhesion 1 such that each bag induces a 2-connected graph and on 2-connected graphs $\mathrm{FO} + \mathrm{conn}_1$ collapses to plain FO. We can then in each bag apply the model checking result for FO on nowhere dense graphs [26] and apply the dynamic programming approach presented in [32] to combine the solutions to a global solution. On the other hand, when we allow conn_2 predicates, there are some simple graph classes that do not exclude a topological minor but have bounded expansion, and on which model checking becomes AW[$\star$]-hard.

After the publication of the conference version of this paper it was also proved that the model checking problem for FO + DP is fixed-parameter tractable on each class excluding a minor [23] and even on each class excluding a topological minor [37], providing an even stronger algorithmic meta-theorem.

It will now be interesting to study other extensions of first-order logic that can express further interesting algorithmic graph problems, such as reachability with regular paths queries. This would, in the simplest case, allow expressing bipartiteness and the odd cycle transversal problem. On the other hand, it is very likely that with general regular paths queries, we will get intractability beyond bounded treewidth graphs. The reason is that with the help of stronger path queries, it may be possible to encode all graphs in grids. By the results of Robertson and Seymour [35], a class has unbounded treewidth if and only if it contains all planar graphs, and in particular all grids as a minor. Hence, an encoding may be possible as soon as the treewidth is unbounded.

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